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Gauged diffeomorphisms and hidden symmetries in Kaluza-Klein theories

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ABSTRACT

We analyze the symmetries that are realized on the massive Kaluza-Klein modes in generic D-dimensional backgrounds with three non-compact directions. For this we construct the unbroken phase given by the decompactification limit, in which the higher Kaluza-Klein modes are massless. The latter admits an infinite-dimensional extension of the three-dimensional diffeomorphism group as local symmetry and, moreover, a current algebra associated to $SL(D-2,\mathbb{R})$ together with the diffeomorphism algebra of the internal manifold as global symmetries. It is shown that the 'broken phase' can be reconstructed by gauging a certain subgroup of the global symmetries. This deforms the three-dimensional diffeomorphisms to a gauged version, and it is shown that they can be governed by a Chern-Simons theory, which unifies the spin-2 modes with the Kaluza-Klein vectors. This provides a reformulation of D-dimensional Einstein gravity, in which the physical degrees of freedom are described by the scalars of a *gauged* non-linear σ-model based on $SL(D-2,\mathbb{R})/SO(D-2)$, while the metric appears in a purely topological Chern-Simons form.

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1 Introduction

Kaluza-Klein theories are currently of decisive importance for modern high-energy physics, not only because of the necessity to make contact between string- or M-theory and phenomenology, but also for the conceptual understanding of string theory in general. For instance, the AdS/CFT correspondence, which provides one of the rare approaches to non-perturbative effects in string theory, requires a Kaluza-Klein analysis of the states appearing on certain AdS backgrounds [1–3]. Second, the connection between 10-dimensional string theory and 11-dimensional supergravity or M-theory relies on a Kaluza-Klein reinterpretation of the spectrum of D0-branes in type IIA [4].

Given this importance of Kaluza-Klein theories it is natural to ask for a detailed understanding of the dynamics of all Kaluza-Klein modes. However, as the main focus was so far mainly on phenomenological applications, most approaches have taken only the lowest or massless modes into account, since the massive modes are supposed to decouple. In contrast, this is not the case in the AdS/CFT correspondence. In fact, the internal manifolds appearing in AdS string backgrounds have to be large [1], and therefore the massive modes can no longer be integrated out. Focusing on the low-energy description, a better understanding of the effective supergravity actions for massive Kaluza-Klein states is therefore desirable.

Recently we initiated in [5] an analysis of the effective gravity actions containing the full tower of Kaluza-Klein modes. These contain an infinite tower of massive spin-2 fields (being the higher modes of the metric) and, in supergravity, an infinite tower of massive spin-3/2 fields as their superpartner. The most natural description of these massive states would be in terms of some spontaneously broken infinite-dimensional (super-)symmetry, which in turn generalizes the super-Higgs effect known from supergravity. Thus the focus of [5] has been on the underlying symmetries of Kaluza-Klein theories.

Based on a circle compactification of pure gravity it has been argued some time ago in [6], that there is indeed an infinite-dimensional spontaneously broken gauge symmetry hidden in the full Kaluza-Klein theory. This infinite-dimensional spin-2 symmetry appears as a remnant of the higher-dimensional diffeomorphism group. More specifically, every diffeomorphism generated by a vector field ξ^M gives, upon Fourier expansion, rise to an infinite-dimensional spin-2 symmetry parametrized by $\xi^{\mu n}$ (with n denoting the Fourier modes) as well as an infinite-dimensional gauge symmetry generated by ξ^{5n}. The latter appears as an ordinary Yang-Mills gauge symmetry, whose Lie algebra is given by the Virasoro algebra, i.e. by the diffeomorphism algebra of the internal manifold (the circle) [7–11]. The former spin-2 symmetries, on the other hand, have been further elaborated in [5] in the case of a Kaluza-Klein reduction to $2+1$ dimensions. They are required in order to guarantee consistency of the gravity–spin-2 couplings in the same sense that supersymmetry is required for consistency of gravity–spin-3/2 couplings.

In addition it has been shown that these theories can be derived ab initio, at least in principle, in a fashion similar to the construction of gauged supergravities [12–15]: It is possible to start from an unbroken phase, in which the spin-2 symmetry is manifestly or linearly realized, while the Virasoro algebra is frozen to a rigid symmetry. This unbroken phase corresponds to the decompactification limit and can formally be determined simply be restoring in the zero-mode action the dependence of all fields on the internal coordinate. In more mathematical terms this means to replace the metric and all matter fields by an 'algebra-valued' object, where the (commutative and associative) algebra is given by the algebra of smooth functions on the circle. This fits into a non-standard form of general relativity introduced by Wald [16,17], which is based on a so-called algebra-valued differential geometry, and which is essentially the only way to get a multi-graviton theory that is consistent with a generalized diffeomorphism or spin-2 symmetry [18–20]. Furthermore, upon performing duality transformations in $D = 3$ it has been shown that the hidden symmetry realized on the zero-modes (the Ehlers group $SL(2,\mathbb{R})$) gets enhanced to its affine extension. The full (massive) Kaluza-Klein theory can then be reconstructed by gauging a certain subalgebra of this rigid symmetry, i.e. by promoting this subalgebra to a local symmetry. This in turn deforms the diffeomorphisms in the sense that, at least, each partial derivative in the standard formulas for diffeomorphisms gets replaced by a covariant derivative D_μ. Due to the non-commutativity $[D_\mu, D_\nu] \sim F_{\mu\nu}$ this turns to a symmetry which is no longer manifest. However, in the (2+1)-dimensional context the spin-2 fields and the gauge vectors combine into a Chern-Simons theory [21–23] based on a Lie algebra which contains a non-standard semi-direct product between the Virasoro algebra and a centrally extended Kac-Moody algebra associated to the Poincaré group. This allows a direct investigation of the gauged diffeomorphisms, due to which the compactification to $D = 3$ is an excellent arena for the analysis of these symmetries.

One aim of the present paper is to show that these results extend to more general Kaluza-Klein backgrounds. We will focus again on compactifications to $D = 3$,[1] but we will also argue that the results are to some extend independent of the internal manifold. We will show that the Virasoro algebra and the (affine) Kac-Moody algebras appearing in S^1 compactifications are replaced by the diffeomorphism algebra of the internal manifold (whose form we will give explicitly in case of a torus) and the Lie algebra of so-called current groups. The latter generalize the affine algebras as the Lie algebras of a Loop group $C^\infty(S^1, G)$ associated to a Lie group G to $C^\infty(K, G)$ for arbitrary compact manifolds K. These are substantially more intricate than Loop groups and so have not been studied exhaustively in the mathematical literature [25, 26]. Investigations of Kaluza-Klein theories might therefore also be of interest in this respect.

A second motivation for the present analysis is to give a reformulation of D-dimensional Einstein gravity in a form that may shed light on the role that the so-called 'hidden symmetries' encountered in dimensional reductions play in the original theory [27–29]. In fact, once all massive Kaluza-Klein modes are taken into account, the Kaluza-Klein theory can still be viewed as being D-dimensional, but in a particular – Kaulza-Klein inspired – Lorentz gauge. This approach has been pioneered in [30, 31], where it has been shown that part of the hidden symmetries appearing via reducing 11-dimensional supergravity to $D = 4$ and $D = 3$, respectively, can be seen already in 11 dimensions, upon fixing part of the Lorentz symmetry. More specifically, the composite local symmetry groups appearing in the coset spaces $E_{7(7)}/SU(8)$ and $E_{8(8)}/SO(16)$ exist also in 11-dimensional supergravity, while the role of the exceptional groups remains somewhat mysterious. (See, however, [32].) In a similar spirit it has been suggested that gravity in D dimensions should have an interpretation as a non-linear σ-model based on $SL(D-2,\mathbb{R})/SO(D-2)$, which is exactly the structure that appears by reducing to $D = 3$ on a torus. It might therefore be of interest that, as we are going to show in this

[1]Kaluza-Klein compactifications to $D = 4$ without truncation have been considered in [24].

paper, gravity in any dimension can be seen as a *gauged* non-linear σ-model of this type (in a sense that we will make precise below). More specifically, the ungauged theory, still being fully D-dimensional, admits the entire $SL(D-2,\mathbb{R})$ as symmetry group, whose breaking in the full theory is due to the gauging. This should be compared with [33], in which $SL(D-2,\mathbb{R})$ has been realized as a symmetry of D-dimensional gravity in light-cone gauge, but with the action in a non-local form.

The paper is organized as follows. After briefly reviewing the structure of Kaluza-Klein theories and symmetries for compactifications to $D=3$ in sec. 2, we discuss in sec. 3 the unbroken phase together with its symmetries. In sec. 4 we turn to the problem of reconstructing the full D-dimensional gravitational theory, i.e. the 'broken phase', via gauging certain symmetries. The appearing consistency problems are discussed, and it is shown that their resolution can be made manifest within a particular subsector given by a Chern-Simons description. We conclude in sec. 5. Appendix A reviews the emergence of the hidden symmetry $SL(D-2,\mathbb{R})$ in torus reductions to $D=3$, while appendix B shows the details of an explicit Kaluza-Klein analysis without truncation.

2 Kaluza-Klein theory on $\mathbb{R}^{1,2}\times K_d$

In this section we give a brief review of Kaluza-Klein theories on backgrounds of the form $\mathbb{R}^{1,2}\times K_d$, where K_d is a priori an arbitrary compact manifold. Even though a non-Ricci-flat K_d would require matter-couplings in the higher-dimensional theory in order for this background to be a solution of the equations of motion, we will focus on the reduction of the Einstein-Hilbert term only since due to its non-linearities it is the most intricate one. Whenever we consider the action and symmetry transformation in terms of a mode expansion, we will specialize to a d-dimensional torus. This should, however, not be confused with a particular truncation, since we will keep the dependence on all $D=3+d$ coordinates.

Our starting point is pure gravity in D dimensions, described by the Einstein-Hilbert action

$$S_{\rm EH}=-\int d^{3+d}x\;ER\;. \tag{2.1}$$

We make a Kaluza-Klein ansatz by fixing the Lorentz symmetry, so that the vielbein appears in a triangular gauge:[2]

$$E_M^{\ A}=\begin{pmatrix}\phi^{-1}e_\mu^{\ a} & A_\mu^{\ m}\phi_m^{\ \alpha}\\ 0 & \phi_m^{\ \alpha}\end{pmatrix}\;. \tag{2.2}$$

Here $\phi_m^{\ \alpha}$ are scalar fields, of which we may think as parametrizing the vielbein of the internal manifold, and

$$\phi=\det(\phi_m^{\ \alpha})=\frac{1}{d!}\epsilon^{m_1\dots m_d}\epsilon_{\alpha_1\dots\alpha_d}\phi_{m_1}^{\ \alpha_1}\dots\phi_{m_d}^{\ \alpha_d}\;. \tag{2.3}$$

[2]Our conventions are as follows: The coordinates are $x^M=(x^\mu,\hat{y}^m)=(x^\mu,gy^m)$. Space-time and Lorentz indices are labelled in D dimensions by $M,N,K,\dots$ and $A,B,C,\dots$, in $2+1$ dimensions by $\mu,\nu,\rho,\dots$ and $a,b,c,\dots$, and finally for the internal d dimensions by $m,n,k,\dots$ and $\alpha,\beta,\gamma,\dots$, respectively. The metrics are mostly minus.

The inverse vielbein reads

$$E_A^M = \begin{pmatrix} \phi e_a^\mu & -e_a^\rho A_\rho^m \phi \\ 0 & \phi_\alpha^m \end{pmatrix} \,, \tag{2.4}$$

where ϕ_α^m denotes the inverse of ϕ_m^α.

The dimensionally reduced action or, equivalently, the zero-mode action in case of a torus, takes the following form

$$\begin{aligned} S = \int d^3x\, e\big[-R^3(e) - \frac{1}{4}\phi^2 G_{mn}(\phi) F^{\mu\nu\, m} F^n_{\mu\nu} + \phi^{-2}\partial^\mu\phi\partial_\mu\phi \\ + \frac{1}{2} g^{\mu\nu}(\phi_\alpha^m \partial_\mu \phi_m^\gamma)(\phi_\gamma^n \partial_\nu \phi_n^\alpha) - \frac{1}{2} G^{mn}(\phi) g^{\mu\nu} \partial_\mu \phi_m^\beta \partial_\nu \phi_{n\beta}\big] \,. \end{aligned} \tag{2.5}$$

Here the gauge kinetic couplings are defined by $G_{mn} = \delta_{\alpha\beta}\phi_m^\alpha \phi_n^\beta$. After this truncation the only remnant of the D-dimensional diffeomorphisms are 3-dimensional diffeomorphisms and $U(1)^d$ gauge tranformations, for which $F_{\mu\nu}^m = \partial_\mu A_\nu^m - \partial_\nu A_\mu^m$ provides the invariant field strength.

Let us next analyze which form the full D-dimensional diffeomorphisms take in the Kaluza-Klein gauge (2.2), or in other words, which symmetry is realized on the full tower of Kaluza-Klein modes without truncation. The D-dimensional diffeomorphisms and local Lorentz transformations are parametrized by ξ^M and $\Lambda^A{}_B$, respectively, and read

$$\delta_\xi E_M^A = \xi^N \partial_N E_M^A + \partial_M \xi^N E_N^A \,, \qquad \delta_\Lambda E_M^A = \Lambda^A{}_B E_M^B \,. \tag{2.6}$$

Splitting the diffeomorphisms as $\xi^M = (\xi^\mu, \xi^m)$, they act on the Kaluza-Klein fields according to

$$\begin{aligned} \delta_\xi \phi_m^\alpha &= \xi^\rho \partial_\rho \phi_m^\alpha + g\xi^n \partial_n \phi_m^\alpha + g\partial_m \xi^\rho A_\rho^n \phi_n^\alpha + g\partial_m \xi^n \phi_n^\alpha \,, \\ \delta_\xi \phi &= \xi^\rho \partial_\rho \phi + g\xi^n \partial_n \phi + g\partial_m \xi^\rho A_\rho^m \phi + g\partial_m \xi^m \phi \,, \\ \delta_\xi A_\mu^m &= \xi^\rho \partial_\rho A_\mu^m + g\xi^n \partial_n A_\mu^m + \partial_\mu \xi^\rho A_\rho^m + \partial_\mu \xi^m - gA_\mu^n \partial_n \xi^\rho A_\rho^m - gA_\mu^n \partial_n \xi^m \,, \\ \delta_\xi e_\mu^a &= \xi^\rho \partial_\rho e_\mu^a + g\xi^m \partial_m e_\mu^a + \partial_\mu \xi^\rho e_\rho^a + gA_\rho^m \partial_m \xi^\rho e_\mu^a + g\partial_m \xi^m e_\mu^a \,. \end{aligned} \tag{2.7}$$

Here we have introduced the radii $R =: g^{-1}$ of the torus which, for simplicity of notation, we take to be equal. In the following the inverse radius g will serve as gauge coupling constant. The transformations (2.7) are in general not compatible with the triangular gauge in (2.2). Thus we have to add a compensating Lorentz transformation, which turns out to be given by

$$\Lambda^a{}_\alpha = -g\phi^{-1}\phi_\alpha{}^m \partial_m \xi^\rho e_\rho^a \,. \tag{2.8}$$

This yields

$$\delta_\Lambda \phi_m^\alpha = 0 \,, \quad \delta_\Lambda e_\mu^a = -gA_\mu^m \partial_m \xi^\rho e_\rho^a \,, \quad \delta_\Lambda A_\mu^m = -g\phi^{-2} G^{mn} \partial_n \xi^\rho g_{\rho\mu} \,. \tag{2.9}$$

Next one can perform a mode expansion associated to a torus, which reads for the scalar, e.g.,

$$\phi(x^\mu, y^m) = \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_d=-\infty}^{\infty} \phi^{[n_1,\dots,n_d]}(x) e^{in_1 \hat{y}^1} \dots\, e^{in_d \hat{y}^d} \,, \tag{2.10}$$

and similarly for all other fields. Moreover, we have to impose a reality constraint on the fields, as $(\phi^{[n_1,...,n_d]})^* = \phi^{[-n_1,...,-n_d]}$. Also the transformation parameter can be expanded into Fourier modes. This results then in an infinite-dimensional symmetry, which is spontaneously broken to the symmetry of the zero-modes [6]. The global symmetry and its Lie algebra in the unbroken phase will be determined in the next section.

3 The unbroken phase and its symmetries

As explained in the introduction, the unbroken phase, in which the spin-2 symmetries are manifestly and linearly realized, can essentially be reconstructed simply by restoring the dependence on the internal coordinates y^m of all fields in the dimensionally reduced action (2.5). This results in

$$\begin{aligned}S_0 = \int d^3x d^dy\, e\big[-R^3(e) - \frac{1}{4}\phi^2 G_{mn}(\phi) F^{\mu\nu\, m} F^n_{\mu\nu} + \phi^{-2}\partial^\mu\phi\partial_\mu\phi \\ + \frac{1}{2} g^{\mu\nu}(\phi^m_\alpha \partial_\mu \phi^\gamma_m)(\phi^n_\gamma \partial_\nu \phi^\alpha_n) - \frac{1}{2} G^{mn}(\phi) g^{\mu\nu} \partial_\mu \phi^\beta_m \partial_\nu \phi_{n\beta}\big] \,.\end{aligned} \tag{3.11}$$

As in contrast to (2.5) the fields depend on all D coordinates the action contains also an integration over the additional d internal coordinates. In this sense (3.11) describes a truly D-dimensional theory, but without the full D-dimensional diffeomorphism and Lorentz invariance. As we are going to show in the following, this theory might instead be viewed as the Kaluza-Klein theory in the decompactification limit $R \to \infty$ (i.e. $g \to 0$), with all Kaluza-Klein modes retained.

It can in turn be seen that, compared to (2.5), in (3.11) an enhancement of the three-dimensional diffeomorphism symmetry takes place. More precisely, with the standard formulas for diffeomorphisms it can be easily checked that they leave (3.11) invariant also if the transformation parameter ξ^μ is allowed to depend on the internal coordinates. Explicitly, they act on the fields as (2.7) with $\xi^m = 0$, $g = 0$.

Besides this local infinite-dimensional diffeomorphism or spin-2 symmetry, there appears also an infinite-dimensional global symmetry group. The latter is the rigid remnant of the diffeomorphism group of the internal manifold. They act like the ξ^m-variations in (2.7), but with ξ^m being independent of space-time, and it can be easily checked that they leave (3.11) invariant. In contrast to the compactification on a circle [5], this group is no longer defined by the Virasoro or Witt algebra, but instead by a more complicated algebra. More specifically, in case that the internal manifold is a torus, the algebra, which we will denote in the following by $\hat{v}_d$, is spanned by generators $Q^{m[j_1,...,j_d]}$ and reads

$$[Q^{m[j_1,...,j_d]}, Q^{n[k_1,...,k_d]}] = i\left(j_n Q^{m[j_1+k_1,...,j_d+k_d]} - k_m Q^{n[j_1+k_1,...,j_d+k_d]}\right) \,. \tag{3.12}$$

The subalgebra spanned by all generators of the form $Q^m_j := Q^{j[m,...,m]}$, where $j = 1, ..., d$, takes the form

$$[Q^m_j, Q^n_j] = i(m-n) Q^{m+n}_j \,, \tag{3.13}$$

and thus the algebra contains, as expected, d copies of the Virasoro algebra. Note, however, that it is not a direct sum, since the Q^m_j do not commute for different j.

As one of the results of [5] it has been found that upon dualization the theory can equivalently be written in a form that admits moreover the affine extension of the hidden symmetry group $SL(2,\mathbb{R})$ (the Ehlers group) as global symmetry group. In general, for reductions on d-dimensional tori to $D=3$ a hidden $SL(d+1,\mathbb{R})$ symmetry appears. (See appendix A for a review.) Thus one might expect that in the unbroken phase also the latter extends to a symmetry on the full Kaluza-Klein tower and has moreover an infinite-dimensional extension. In the following we are going to show that this is indeed the case.

The corresponding action can equally be determined from the zero-mode action in the form (A.3), where all degrees of freedom have been dualized into scalars:

$$
\begin{aligned}
S_0 = \int d^3x d^dy\, e\big[- R^3(e) + g^{\mu\nu}(\phi^{-2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\phi^{-2}G^{mn}(\phi)\partial_\mu\varphi_m\partial_\nu\varphi_n \\
+ \frac{1}{2}\phi_\alpha^m\partial_\mu\phi_m^\gamma\phi_\gamma^n\partial_\nu\phi_n^\alpha - \frac{1}{2}G^{mn}(\phi)\partial_\mu\phi_m^\beta\partial_\nu\phi_{n\beta})\big] \ .
\end{aligned}
\tag{3.14}
$$

Here the dual scalars φ_m are defined by means of the duality relation (A.2) given in appendix A, but again with all fields depending also on the y^m. The $SL(d+1,\mathbb{R})$ symmetry transformations given in appendix A can then depend also on the internal coordinates without affecting the invariance of the action. The underlying Lie algebra is, however, more complicated than the affine, that is, loop group extension which appears in case of an S^1 compactification. Instead the group is given by the smooth maps from the compact manifold K_d into the considered Lie group, $C^\infty(K_d, SL(d+1,\mathbb{R}))$. In the mathematical literature these are known as current groups. In case that the compact manifold is given by the torus T^d its Lie algebra will be denoted in the following by $T^d sl(d+1,\mathbb{R})$.[3] Like in the case of affine Kac-Moody algebras, the algebra $T^d\mathfrak{g}$ associated to any finite-dimensional Lie algebra $\mathfrak{g}$ can essentially be determined by endowing the generators of $\mathfrak{g}$ with the exponential Fourier modes as in (2.10). Let us illustrate this for $sl(d+1,\mathbb{R})$. In the basis defined in appendix A the generators are given by $K^a{}_b$ $(a,b=1,...,d)$, which span the $sl(d,\mathbb{R})$ subalgebra, as well as e_a, f^a and $\hat{e}$, whose transformation properties under $sl(d,\mathbb{R})$ are given in (A.8). The current subalgebra $T^d sl(d,\mathbb{R})$ can then simply be read off from (A.4),

$$
[K^{a[j_1,...,j_d]}_{\ \ b}, K^{c[k_1,...,k_d]}_{\ \ d}] = \delta^a_d K^{c[j_1+k_1,...,j_d+k_d]}_{\ \ b} - \delta^c_b K^{a[j_1+k_1,...,j_d+k_d]}_{\ \ d} \ , \tag{3.15}
$$

while for the remaining brackets one finds from (A.8)

$$
\begin{aligned}
&[K^{a[\underline{j}]}_{\ \ b}, e_c^{[\underline{k}]}] = \delta^a_c e_b^{[\underline{j}+\underline{k}]} - \frac{1}{d}\delta^a_b e_c^{[\underline{j}+\underline{k}]} \ , \qquad [K^{a[\underline{j}]}_{\ \ b}, f^{c[\underline{k}]}] = -\delta^c_b f^{a[\underline{j}+\underline{k}]} + \frac{1}{d}\delta^a_b f^{c[\underline{j}+\underline{k}]};, \\
&[e_a^{[\underline{j}]}, f^{b[\underline{k}]}] = K^{b[\underline{j}+\underline{k}]}_{\ \ a} - \frac{1}{d}\hat{e}^{[\underline{j}+\underline{k}]}\delta^b_a \ , \quad [K^{a[\underline{j}]}_{\ \ b}, \hat{e}^{[\underline{k}]}] = 0 \ . \\
&[e_a^{[\underline{j}]}, \hat{e}^{[\underline{k}]}] = (d+1)e_a^{[\underline{j}+\underline{k}]} \ , \qquad [f^{a[\underline{j}]}, \hat{e}^{[\underline{k}]}] = -(d+1)f^{a[\underline{j}+\underline{k}]} \ , \\
&[e_a^{[\underline{j}]}, e_b^{[\underline{k}]}] = 0 \ , \qquad [f_a^{[\underline{j}]}, f_b^{[\underline{k}]}] = 0 \ .
\end{aligned}
\tag{3.16}
$$

Here we have introduced the compact notation $Q^{m[\underline{j}]}$, etc., where $[\underline{j}]$ denotes the row vector $[j_1,...,j_d]$.

[3] In the case $d=1$ this reduces to the affine extension of $SL(2,\mathbb{R})$ discussed in [5].

To find the full rigid symmetry algebra in the unbroken phase, we have to ask if there is some non-trivial product between the diffeomorphism algebra $\hat{v}_d$ and the current algebra associated to $SL(d+1,\mathbb{R})$. In the case of a circle compactification this reduces – upon a simple change of basis – to the standard form of a semi-direct product between a Virasoro algebra and a Kac-Moody algebra [5], which is well-known to physicists from the Sugawara construction. In the more general case, however, the resulting structure is not clear a priori and therefore will be analyzed in the following.

First of all, we have to know the action of the internal diffeomorphism algebra $\hat{v}_d$ on all physical fields, in particular on the dual scalar φ_m (carrying the former degrees of freedom of the Kaluza-Klein vectors A_μ^m). This can be determined by applying the ξ^m variations in (2.7) to the duality relation (A.2). One finds after some computations[4]

$$\delta_\xi \varphi_m = \xi^k \partial_k \varphi_m + \partial_k \xi^k \varphi_m + \partial_m \xi^k \varphi_k \ . \tag{3.17}$$

In order to determine the 'semi-direct' product let us first check the closure of the symmetry variations (with y^m-dependent parameter). One finds

$$\begin{aligned}
[\delta_\xi(Q), \delta_\lambda(K)]\phi_m^\alpha &= \delta_{\tilde{\lambda}}(K)\phi_m^\alpha \ , \\
[\delta_\xi(Q), \delta_\kappa(e)]\varphi_m &= \delta_{\tilde{\kappa}}(e)\varphi_m \ , \\
[\delta_\xi(Q), \delta_\sigma(f)]\phi_m^\alpha &= \delta_{\tilde{\sigma}}(f)\phi_m^\alpha \ , \\
[\delta_\xi(Q), \delta_\epsilon(\hat{e})]\phi_m^\alpha &= \delta_{\tilde{\epsilon}}(\hat{e})\phi_m^\alpha \ ,
\end{aligned} \tag{3.18}$$

where the transformation parameter are given by

$$\begin{aligned}
\tilde{\lambda}^k{}_m &= -\xi^n \partial_n \lambda^k{}_m + \lambda^n{}_m \partial_n \xi^k - \partial_m \xi^n \lambda^k{}_n \ , \\
\tilde{\kappa}_m &= -\xi^k \partial_k \kappa_m - \partial_k \xi^k \kappa_m - \partial_m \xi^k \kappa_k \ , \\
\tilde{\sigma}^m &= -\xi^k \partial_k \sigma^m + \partial_k \xi^k \sigma^m + \sigma^k \partial_k \xi^m \ , \\
\tilde{\epsilon} &= -\xi^k \partial_k \epsilon \ .
\end{aligned} \tag{3.19}$$

By expanding (3.19) into Fourier modes one can read off the Lie algebra:

$$\begin{aligned}
[Q^{m[\underline{j}]}, K^{n[\underline{k}]}_{\ \ p}] &= -ik_m K^{n[\underline{j}+\underline{k}]}_{\ \ p} + ij_p K^{n[\underline{j}+\underline{k}]}_{\ \ m} - ij_q \delta^{mn} K^{q[\underline{j}+\underline{k}]}_{\ \ p} \ , \\
[Q^{p[\underline{j}]}, f_q^{[\underline{k}]}] &= i\big(-k_p f_q^{[\underline{j}+\underline{k}]} + j_p f_q^{[\underline{j}+\underline{k}]} + j_q f_p^{[\underline{j}+\underline{k}]} \big) \ , \\
[Q^{p[\underline{j}]}, e^{q[\underline{k}]}] &= -i\left((k_p + j_p) e^{q[\underline{j}+\underline{k}]} + \delta^{pq} j_l (e^l)^{[\underline{j}+\underline{k}]} \right) \ , \\
[Q^{n[\underline{j}]}, \hat{e}^{[\underline{k}]}] &= -ik_n \hat{e}^{[\underline{j}+\underline{k}]} \ .
\end{aligned} \tag{3.20}$$

Note that this Lie algebra reduces for $d = 1$ to the algebra of [5], if one identifies (in the notation of [5]) h with $\hat{e}$, f with f_1 and e with e^1, while $K^a{}_b$ trivializes (in order to be traceless).

One can check explicitly that this defines a consistent Lie algebra satisfying the Jacobi identities, generalizing the well-known semi-direct product between the standard Virasoro algebra and an affine Kac-Moody algebra.

[4]Here we have rescaled the transformation parameter such that in the ungauged phase no factor of g appears.

4 Reconstruction of D-dimensional Einstein gravity

In the last section we have shown that the dual action (3.14) is invariant under an infinite-dimensional diffeomorphism symmetry and an infinite-dimensional extension of the hidden rigid invariance group $SL(d+1,\mathbb{R})$. It is remarkable that a truly D-dimensional action still admits the $SL(D-2,\mathbb{R})$ symmetry. However, the diffeomorphism group of the internal manifold is also realized only as a global symmetry on the full Kaluza-Klein tower, or equivalently, local only in the internal coordinates.

As a next step we are going to reconstruct the full D-dimensional gravity theory. For this we have to promote, at least, the internal diffeomorphism algebra to a local symmetry, i.e. we have to gauge the subalgebra $\hat{v}_d$. In the next subsection we are going to determine the covariant derivatives, which are required for a minimal coupling, and we will briefly discuss the appearing consistency problems related to 'gauged diffeomorphisms'. Then we discuss the manifest resolution of these consistency problems for a particular subsector of the theory, namely the sector consisting of gravitational and gauge fields. Finally we give the full action, which is on-shell equivalent to the original D-dimensional gravity theory.

4.1 Covariantisation and gauged diffeomorphisms

To begin with, we have to replace all partial derivatives by covariant derivatives with respect to $\hat{v}_d$. These can be most conveniently written by adding additional terms proportional to the internal derivative ∂_m. They read

$$\begin{aligned}
D_\mu\phi_m^\alpha &= \partial_\mu\phi_m^\alpha - gA_\mu^n\partial_n\phi_m^\alpha - g\phi_n^\alpha\partial_m A_\mu^n \ , \\
D_\mu\phi &= \partial_\mu\phi - gA_\mu^n\partial_n\phi - g\phi\partial_n A_\mu^n \ , \\
D_\mu\varphi_m &= \partial_\mu\varphi_m - gA_\mu^n\partial_n\varphi_m - g\varphi_n\partial_m A_\mu^n - g\varphi_m\partial_k A_\mu^k \ , \\
D_\mu e_\nu^a &= \partial_\mu e_\nu^a - gA_\mu^m\partial_m e_\nu^a - ge_\nu^a\partial_m A_\mu^m \ , \\
D_\mu\omega_\nu^a &= \partial_\mu\omega_\nu^a - gA_\mu^m\partial_m\omega_\nu^a \ .
\end{aligned} \tag{4.21}$$

Here ω_μ^a denotes the spin connection, which transforms under $\hat{v}_d$ as

$$\delta_\xi\omega_\mu^a = g\xi^m\partial_m\omega_\mu^a \ . \tag{4.22}$$

One can easily check that the covariant derivatives (4.21) transform covariantly under local $\hat{v}_d$ transformations, e.g.

$$\delta_{\xi^m}(D_\mu\phi_m^\alpha) = g\xi^n\partial_n(D_\mu\phi_m^\alpha) + g\partial_m\xi^n D_\mu\phi_n^\alpha \ . \tag{4.23}$$

Comparing with (2.7) we see that $D_\mu\phi_m^\alpha$ transforms exactly as ϕ_m^α and similarly for all other fields. Thus, at this stage $\hat{v}_d$ is manifestly realized.

Apart from the problem that we did not yet introduce a kinetic term for the $\hat{v}_d$ gauge fields A_μ^m, we have to ask the question if the spin-2 transformations or generalized diffeomorphisms parametrized by ξ^μ are still a symmetry. This is not the case, simply due to the fact that the internal derivatives ∂_m in the covariant derivatives also act on the spin-2 transformation parameter ξ^ρ. Put differently, the covariant derivative of, say, a scalar, will not transform like a 1-form, but will pick up a non-covariant piece. In order to keep spin-2 invariance we also have to deform the diffeomorphisms by g-dependent terms. Their actual form can partially be determined by requiring closure of local $\hat{v}_d$ with spin-2 transformations. For the scalars this implies

$$\delta_\xi \phi_m^\alpha = \xi^\rho \partial_\rho \phi_m^\alpha + g\phi_n^\alpha \partial_m \xi^\rho A_\rho^n \ , \tag{4.24}$$

since then the algebra closes according to

$$[\delta_\xi, \delta_\eta(Q)]\phi_m^\alpha = \delta_{\tilde{\xi}} \phi_m^\alpha + \delta_{\tilde{\eta}}(Q)\phi_m^\alpha \ , \tag{4.25}$$

where the transformation parameter are given by

$$\tilde{\xi}^\rho = \eta^n \partial_n \xi^\rho \ , \qquad \tilde{\eta}^n = -\xi^\rho \partial_\rho \eta^n \ . \tag{4.26}$$

Comparing with (2.7) one infers that requiring closure of the symmetry variations allows to recover the expected spin-2 transformations for ϕ_m^α. Similarly, one shows that for e_μ^a the spin-2 variations get deformed in the gauged phase according to (2.7). However, the variations for A_μ^m determined like this do not coincide completely with (2.7) and (2.9), since the term depending on ϕ is not necessary for the closure as in (4.25) and (4.26). We will come back to this point later. For the dual scalars φ_m one finds correspondingly

$$\delta_\xi \varphi_m = \xi^\rho \partial_\rho \varphi_m + 2g\varphi_{(m}\partial_{n)}\xi^\rho A_\rho^n \ . \tag{4.27}$$

Let us now check if the covariant derivatives (4.21) transform covariantly also under the deformed spin-2 transformations. For this it will prove to be convenient to consider a particular combination of a local $\hat{v}_d$ transformation and a spin-2 transformation. We consider a diffeomorphism generated by ξ^μ and add a local $\hat{v}_d$ transformation with field-dependent parameter $\xi^m = -\xi^\rho A_\rho^m$. This results in

$$\begin{aligned} \delta_\xi \phi_m^\alpha &= \xi^\rho D_\rho \phi_m^\alpha \ , \qquad & \delta_\xi \varphi_m &= \xi^\rho D_\rho \varphi_m \ , \\ \delta_\xi e_\mu^a &= \xi^\rho D_\rho e_\mu^a + D_\mu \xi^\rho e_\rho^a \ , \quad & \delta_\xi A_\mu^m &= \xi^\rho F_{\rho\mu}^m \ , \end{aligned} \tag{4.28}$$

where we have introduced a covariant derivative on the transformation parameter ξ^ρ,

$$D_\mu \xi^\rho = \partial_\mu \xi^\rho - gA_\mu^m \partial_m \xi^\rho \ . \tag{4.29}$$

We see that except the gauge field A_μ^m the fields transform like covariant tensors under a gauged notion of diffeomorphisms, where all partial derivatives have been replaced by covariant derivatives with respect to $\hat{v}_d$. Moreover, an action which is constructed out of a density that transforms accordingly under gauged diffeomorphisms,

$$\delta_\xi(e\mathcal{L}) = D_\rho(e\xi^\rho \mathcal{L}) \ , \tag{4.30}$$

is invariant if and only if it is also invariant under local $\hat{v}_d$ transformations. This can be shown in complete analogy to [5]. Unfortunately, in contrast to the 'bare' fields, the covariant derivatives of the latter do not transform covariantly under these gauged diffeomorphisms. This is due to the non-commutativity of $\hat{v}_d$ covariant derivatives. In fact, it can be easily checked that

$$\begin{aligned} [D_\mu, D_\nu]\phi^\alpha_m &= -g\partial_n\phi^\alpha_m F^n_{\mu\nu} - g\phi^\alpha_n\partial_m F^n_{\mu\nu} \ , \\ [D_\mu, D_\nu]e^a_\rho &= -gF^n_{\mu\nu}\partial_n e^a_\rho - ge^a_\rho\partial_m F^m_{\mu\nu} \ . \end{aligned} \tag{4.31}$$

With these relations it can be shown that, e.g.,

$$\delta_\xi(D_\mu\phi^\alpha_m) = \xi^\rho D_\rho(D_\mu\phi^\alpha_m) + D_\mu\xi^\rho D_\rho\phi^\alpha_m - g\phi^\alpha_n\partial_m\xi^\rho F^n_{\rho\mu} \ , \tag{4.32}$$

i.e. it appears an additional term proportional to the field strength. This in turn implies that generic $\hat{v}_d$-covariant actions will not be invariant under the deformed (or gauged) diffeomorphisms. On the other hand, Kaluza-Klein theories provide by construction a resolution of these consistency problems. We will see that they can indeed be written in a manifestly $\hat{v}_d$-covariant form, i.e. with all appearing derivatives being covariant in the sense defined above. At the same time they are invariant under the spin-2 transformations defined in (2.7) and (2.9), the latter fact just expressing the diffeomorphism invariance of the original Einstein-Hilbert action. However, inspecting (2.7) and (2.9) more closely, we infer that this can only be achieved by adding a scalar-dependent term to the variation $\delta_\xi A^m_\mu$. Moreover, we will see that the full action contains additional terms, beyond those resulting just from a minimal substitution in (3.14). Those couplings will contain a scalar potential and spin-2 mass terms, and the total variation of the action under the gauged diffeomorphisms will therefore link all terms in a non-trivial way. This should be compared to the gauging of supergravity, where the supersymmetry variations also have to be extended by terms proportional to the gauge-coupling. So the invariance under gauged diffeomorphisms, which is guaranteed by construction, is not manifest, but we are going to prove in the next section that on the subsector of purely gravitational and gauge fields this invariance can be even made manifest via a Chern-Simons description.

4.2 Chern-Simons theory for gravitational and gauge fields

As we have discussed in the last section, simply covariantising a spin-2 invariant action with respect to the internal diffeomorphism algebra $\hat{v}_d$ in general does not result in an action which is invariant under any covariantiation of the spin-2 symmetries. This drawback holds also for the Einstein-Hilbert term, which in turn is the reason that the spin-2 graviton usually cannot be charged with respect to some gauge group. Generalizing the results of [5] and in analogy to gauged supergravity we are going to show that in the 2+1 dimensional context consistency can be regained by adding a Chern-Simons term for the gauge fields. The pure Einstein-Hilbert term in the ungauged phase – still depending on all D coordinates – has, on the other hand, also an interpretation as a Chern-Simons theory, where the gauge group is given by the current group $T^d ISO(1,2)$ associated to the Poincaré group (see appendix A of [5], which generalizes [21]). Thus, following [5], one might hope to be able to combine these Chern-Simons terms into one Chern-Simons theory for some extended Lie algebra. In this spirit the question of a consistent extension of the 'gauged' Einstein-Hilbert term translates into a purely algebraic problem. Namely, it has to be shown that a consistent Lie algebra exists, which has the following properties: First it has to be a semi-direct product between $T^d iso(1,2)$ and $\hat{v}_d$, which gives rise to the correct transformation properties under $\hat{v}_d$ given in (2.7). Second, there has to exist an invariant non-degenerate quadratic form, which can be used to construct a Chern-Simons action which is gauge-invariant, and whose equations of motion do not degenerate. It turns out that these requirements can indeed be satisfied.

First of all we have to note that for the $\hat{v}_d$ generators $Q^{m[\underline{j}]}$ alone an invariant bilinear form does not exist. Thus, as in [5], we have to extend the Lie algebra further by adding generators $e^{m[\underline{j}]}$, which transform under $\hat{v}_d$ in such a way that the bilinear expression $Q^{m[\underline{j}]}e^{m[-\underline{j}]}$ is invariant. These are exactly given by the 'shift' generators of $T^d sl(d+1,\mathbb{R})$ in (3.20). Put differently, we have to gauge not only $\hat{v}_d$, but the entire subalgebra of the rigid symmetry in sec. 3 which is spanned by $Q^{m[\underline{j}]}$ and $e^{m[\underline{j}]}$.

The Lie algebra consistent with the above requirements reads

$$\begin{aligned}
[P_a^{[\underline{j}]}, J_b^{[\underline{k}]}] &= \varepsilon_{abc} P^{c[\underline{j}+\underline{k}]} + i\alpha k_p \eta_{ab} e^{p[\underline{j}+\underline{k}]} \ , \\
[J_a^{[\underline{j}]}, J_b^{[\underline{k}]}] &= \varepsilon_{abc} J^{c[\underline{j}+\underline{k}]} \ , \qquad [P_a^{[\underline{j}]}, P_b^{[\underline{k}]}] = 0 \ , \\
[Q^{m[\underline{j}]}, Q^{n[\underline{k}]}] &= ig\left(j_n Q^{m[\underline{j}+\underline{k}]} - k_m Q^{n[\underline{j}+\underline{k}]}\right) \ , \\
[Q^{m[\underline{j}]}, P_a^{[\underline{k}]}] &= ig(-j_m - k_m) P_a^{[\underline{j}+\underline{k}]} \ , \\
[Q^{m[\underline{j}]}, J_a^{[\underline{k}]}] &= -igk_m J_a^{[\underline{j}+\underline{k}]} \ , \\
[Q^{m[\underline{j}]}, e^{n[\underline{k}]}] &= -ig\left((j_m + k_m) e^{n[\underline{j}+\underline{k}]} + \delta^{mn} j_l (e^l)^{[\underline{j}+\underline{k}]}\right) \ , \\
[P_a^{m[\underline{j}]}, e^{n[\underline{k}]}] &= [J_a^{m[\underline{j}]}, e^{n[\underline{k}]}] = [e^{m[\underline{j}]}, e^{n[\underline{k}]}] = 0 \ ,
\end{aligned} \tag{4.33}$$

and leaves only one free parameter α. Here P_a and J_a denote the Poincaré generator. We observe like in [5] that the $e^{m[\underline{j}]}$ act as central extensions for the Poincaré subalgebra (or as non-central extensions for the full algebra). As required this algebra carries an invariant non-degenerate bilinear form. In the basis (4.33) it reads

$$\langle P_a^{[\underline{j}]}, J_b^{[\underline{k}]} \rangle = \eta_{ab} \delta^{[\underline{j}],[\underline{k}]} \ , \qquad \langle Q^{m[\underline{j}]}, e^{n[\underline{k}]} \rangle = \frac{g}{\alpha} \delta^{mn} \delta^{[\underline{j}],[\underline{k}]} \ , \tag{4.34}$$

while all other terms vanish. Here we have introduced the short-hand notation $\delta^{[\underline{j}],[\underline{k}]} = \delta^{j_1,k_1} \dots \delta^{j_d,k_d}$. Note that the invariance of this quadratic form is only insured due to the central extension of the Poincaré subalgebra.

As a next step we are going to construct the Chern-Simons action associated to the Lie algebra (4.33). The action for a Lie algebra valued gauge field $\mathcal{A}_\mu$ is given by

$$\begin{aligned}
S_{\text{CS}} &= \int \text{Tr}\big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\big) \\
&= \frac{1}{2} \int d^3x \, \varepsilon^{\mu\nu\rho} \big(\langle \mathcal{A}_\mu, \partial_\nu \mathcal{A}_\rho - \partial_\rho \mathcal{A}_\nu \rangle + \frac{2}{3} \langle \mathcal{A}_\mu, [\mathcal{A}_\nu, \mathcal{A}_\rho] \rangle\big) \ ,
\end{aligned} \tag{4.35}$$

where the trace is a symbolic notation for an invariant quadratic form, used explicitly in the second line of (4.35). Writing the gauge field as

$$\mathcal{A}_\mu = e_\mu^{a[\underline{j}]} P_a^{[\underline{j}]} + \omega_\mu^{a[\underline{j}]} J_a^{[\underline{j}]} + A_\mu^{m[\underline{j}]} Q^{m[\underline{j}]} + B_{\mu m}^{[\underline{j}]} e^{m[\underline{j}]} \ , \tag{4.36}$$

and inserting into (4.35) gives by use of (4.33) and (4.34) rise to the following action

$$S_{\rm CS}=\int d^3x\,\varepsilon^{\mu\nu\rho}\big(e_\mu^{a[-\underline{j}]}(D_\nu\omega_{\rho a}^{[\underline{j}]}-D_\rho\omega_{\nu a}^{[\underline{j}]}+\epsilon^{abc}\omega_\nu^{b[\underline{j}-\underline{k}]}\omega_\rho^{a[\underline{k}]})\big)+\frac{g}{\alpha}\varepsilon^{\mu\nu\rho}B_{\mu m}^{[-\underline{j}]}F_{\nu\rho}^{m[\underline{j}]}\ . \quad (4.37)$$

Thus, we exactly recover the covariantized Einstein-Hilbert term, in which the partial derivatives have been replaced by covariant ones with respect to $\hat{v}_d$. In addition we get a Chern-Simons term for the gauge fields.

Let us now discuss the equations of motion and the symmetries of this Chern-Simons theory. Varying (4.35) with respect to the gauge field $\mathcal{A}_\mu$ yields vanishing field strength as the equations of motion. For (4.37) this implies

$$\mathcal{F}_{\mu\nu}=R_{\mu\nu}^{a[\underline{j}]}J_a^{[\underline{j}]}+T_{\mu\nu}^{a[\underline{j}]}P_a^{[\underline{j}]}+F_{\mu\nu}^{m[\underline{j}]}Q^{m[\underline{j}]}+G_{\mu\nu\,m}^{[\underline{j}]}e^{m[\underline{j}]}=0\ , \quad (4.38)$$

whose components read

$$\begin{aligned}
R_{\mu\nu}^{a[\underline{j}]} &= \partial_\mu\omega_\nu^{a[\underline{j}]}-\partial_\nu\omega_\mu^{a[\underline{j}]}+\varepsilon^{abc}\omega_{\mu b}^{[\underline{j}-\underline{k}]}\omega_{\nu c}^{[\underline{k}]} \qquad (4.39)\\
&\quad +ig(j_n-k_n)\omega_\mu^{a[\underline{j}-\underline{k}]}A_\nu^{n[\underline{k}]}-igk_mA_\mu^{m[\underline{j}-\underline{k}]}\omega_\nu^{a[\underline{k}]}\ ,\\
T_{\mu\nu}^{a[\underline{j}]} &= \partial_\mu e_\nu^{a[\underline{j}]}-\partial_\nu e_\mu^{a[\underline{j}]}+\varepsilon^{abc}e_{\mu b}^{[\underline{j}-\underline{k}]}\omega_{\nu c}^{[\underline{k}]}+\varepsilon^{abc}\omega_{\mu b}^{[\underline{j}-\underline{k}]}e_{\nu c}^{[\underline{k}]}\\
&\quad +igj_ne_\mu^{a[\underline{j}-\underline{k}]}A_\nu^{n[\underline{k}]}-igj_mA_\mu^{m[\underline{j}-\underline{k}]}e_\nu^{a[\underline{k}]}\ ,\\
F_{\mu\nu}^{m[\underline{j}]} &= \partial_\mu A_\nu^{m[\underline{j}]}-\partial_\nu A_\mu^{m[\underline{j}]}+ig(j_n-k_n)A_\mu^{m[\underline{j}-\underline{k}]}A_\nu^{n[\underline{k}]}-igk_nA_\mu^{n[\underline{j}-\underline{k}]}A_\nu^{m[\underline{k}]}\ ,\\
G_{\mu\nu\,m}^{[\underline{j}]} &= \partial_\mu B_{\nu\,m}-\partial_\nu B_{\mu\,m}+i\alpha k_me_\mu^{a[\underline{j}-\underline{k}]}\omega_{\nu a}^{[\underline{k}]}-i\alpha(j_m-k_m)\omega_\mu^{a[\underline{j}-\underline{k}]}e_{\nu a}^{[\underline{k}]}\\
&\quad +igj_nB_{\mu m}^{[\underline{j}-\underline{k}]}A_\nu^{n[\underline{k}]}+igk_mB_{\mu n}^{[\underline{j}-\underline{k}]}A_\nu^{n[\underline{k}]}\\
&\quad -igj_nA_\mu^{n[\underline{j}-\underline{k}]}B_{\nu m}^{[\underline{k}]}-ig(j_m-k_m)A_\mu^{n[\underline{j}-\underline{k}]}B_{\nu n}^{[\underline{k}]}\ .
\end{aligned}$$

Like the covariant derivatives also the field strength given here can be conveniently rewritten by taking the fields to be dependent on all D coordinates and writing the non-abelian contributions in (4.39) by means of an internal derivative ∂_m. This results in

$$\begin{aligned}
R_{\mu\nu}^a &= D_\mu\omega_\nu^a-D_\nu\omega_\mu^a+\varepsilon^{abc}\omega_{\mu b}\omega_{\nu c}\ , \qquad (4.40)\\
T_{\mu\nu}^a &= D_\mu e_\nu^a-D_\nu e_\mu^a+\varepsilon^{abc}e_{\mu b}\omega_{\nu c}+\varepsilon^{abc}\omega_{\mu b}e_{\nu c}\ ,\\
F_{\mu\nu}^m &= \partial_\mu A_\nu^m-\partial_\nu A_\mu^m-gA_\mu^n\partial_nA_\nu^m+gA_\nu^n\partial_nA_\mu^m\ ,\\
G_{\mu\nu\,m} &= \partial_\mu B_{\nu\,m}-\partial_\nu B_{\mu\,m}-2gA_{[\mu}^n\partial_nB_{\nu]m}+2gB_{[\mu n}\partial_mA_{\nu]}^n\\
&\quad +2gB_{[\mu m}\partial_nA_{\nu]}^n+2\alpha e_{[\mu}^a\partial_m\omega_{\nu]a}\ .
\end{aligned}$$

We observe in particular that the standard formulas for the Riemann tensor $R_{\mu\nu}^a$ (in three dimensions) and for the torsion tensor $T_{\mu\nu}^a$ are recovered, but with all derivatives being $\hat{v}_d$ covariant. Let us note that the torsion constraint $T_{\mu\nu}^a=0$ following from the equations of motion (4.38) can be used as in standard gravity to solve for the spin connection ω_μ^a, but here in terms of both e_μ^a and A_μ^m.

In order to analyze the symmetries of (4.37) and thus of (4.39), let us consider the non-abelian gauge transformations determined by (4.33). Even though the Chern-Simons action is not manifestly gauge invariant, it can be easily checked that up to a total derivative it is invariant under the gauge transformations $\delta\mathcal{A}_\mu = D_\mu u = \partial_\mu u + [\mathcal{A}_\mu, u]$ generated by a Lie algebra valued transformation parameter u. Writing the transformation parameter as

$$u = \rho^{a[\underline{j}]} P_a^{[\underline{j}]} + \tau^{a[\underline{j}]} J_a^{[\underline{j}]} + \xi^{n[\underline{j}]} Q^{n[\underline{j}]} + \Lambda_n^{[\underline{j}]} e^{n[\underline{j}]} \,, \tag{4.41}$$

the gauge transformations are given by

$$\begin{aligned}
\delta e_\mu^{a[\underline{j}]} &= \partial_\mu \rho^{a[\underline{j}]} + \varepsilon^{abc} e_{\mu b}^{[\underline{j}-\underline{k}]} \tau_c^{[\underline{j}]} + \varepsilon^{abc} \omega_{\mu b}^{[\underline{j}-\underline{k}]} \rho_c^{[\underline{k}]} \\
&\quad + i g j_n \xi^{n[\underline{k}]} e_\mu^{a[\underline{j}-\underline{k}]} - i g j_m \rho^{a[\underline{k}]} A_\mu^{m[\underline{j}-\underline{k}]} \,, \\
\delta \omega_\mu^{a[\underline{j}]} &= \partial_\mu \tau^{a[\underline{j}]} + \varepsilon^{abc} \omega_{\mu b}^{[\underline{j}-\underline{k}]} \tau_c^{[\underline{k}]} + i g (j_n - k_n) \xi^{n[\underline{k}]} \omega_\mu^{a[\underline{j}-\underline{k}]} - i g k_m \tau^{a[\underline{k}]} A_\mu^{m[\underline{j}-\underline{k}]} \,, \\
\delta A_\mu^{m[\underline{j}]} &= \partial_\mu \xi^{m[\underline{j}]} + i g (j_n - k_m) \xi^{n[\underline{k}]} A_\mu^{m[\underline{j}-\underline{k}]} - i g k_n \xi^{m[\underline{k}]} A_\mu^{n[\underline{j}-\underline{k}]} \,, \\
\delta B_{\mu m}^{[\underline{j}]} &= \partial_\mu \Lambda_m^{[\underline{j}]} + i\alpha k_m e_\mu^{a[\underline{j}-\underline{k}]} \tau_a^{[\underline{k}]} - i\alpha (j_m - k_m) \omega_\mu^{a[\underline{j}-\underline{k}]} \rho_a^{[\underline{k}]} - i g j_n A_\mu^{n[\underline{j}-\underline{k}]} \Lambda_m^{[\underline{k}]} \\
&\quad - i g (j_m - k_m) A_\mu^{n[\underline{j}-\underline{k}]} \Lambda_n^{[\underline{k}]} + i g j_n B_{\mu m}^{[\underline{j}-\underline{k}]} \xi^{n[\underline{k}]} + i g k_m B_{\mu n}^{[\underline{j}-\underline{k}]} \xi^{n[\underline{k}]} \,.
\end{aligned} \tag{4.42}$$

Also the gauge transformations can be conveniently rewritten by taking y-dependent fields and transformation parameters. The result reads

$$\begin{aligned}
\delta e_\mu^a &= \partial_\mu \rho^a + \epsilon^{abc} e_{\mu b} \tau_c + \epsilon^{abc} \omega_{\mu b} \rho_c + g \xi^n \partial_n e_\mu^a + g \partial_n \xi^n e_\mu^a \\
&\quad - g \rho^a \partial_m A_\mu^m - g \partial_m \rho^a A_\mu^m \,, \\
\delta \omega_\mu^a &= \partial_\mu \tau^a + \epsilon^{abc} \omega_{\mu b} \tau_c + g \xi^n \partial_n \omega_\mu^a - g A_\mu^m \partial_m \tau^a \,, \\
\delta A_\mu^m &= \partial_\mu \xi^m + g \xi^n \partial_n A_\mu^m - g A_\mu^n \partial_n \xi^m \,, \\
\delta B_{\mu m} &= \partial_\mu \Lambda_m - g \Lambda_m \partial_n A_\mu^n - g A_\mu^n \partial_n \Lambda_m - g \Lambda_n \partial_m A_\mu^n \\
&\quad + g \xi^n \partial_n B_{\mu m} + g B_{\mu m} \partial_n \xi^n + g \partial_m \xi^n B_{\mu n} + \alpha e_\mu^a \partial_m \tau_a - \alpha \rho_a \partial_m \omega_\mu^a \,.
\end{aligned} \tag{4.43}$$

Let us now check, if the symmetries expected for Kaluza-Klein theories from (2.7) and (2.9) are contained in (4.43). First of all we notice by comparing with (2.7), that the $\hat{v}_d$ gauge transformations parametrized by ξ^m are correctly reproduced. Moreover, (4.43) allows us to compare with the deformed spin-2 transformations, or in other words, to see if one recovers the gauged diffeomorphisms. For this we consider the non-abelian gauge transformations for the field-dependent transformation parameter

$$\rho^a = \xi^\mu e_\mu^a \,, \qquad \tau^a = \xi^\mu \omega_\mu^a \,, \qquad \xi^m = \xi^\mu A_\mu^m \,, \qquad \Lambda_m = \xi^\mu B_{\mu m} \,. \tag{4.44}$$

Then the gauge transformations (4.43) take the following form

$$\begin{aligned}
\delta e_\mu^a &= \xi^\rho \partial_\rho e_\mu^a + \partial_\mu \xi^\rho e_\rho^a + g A_\rho^m \partial_m \xi^\rho e_\mu^a - g A_\mu^m \partial_m \xi^\rho e_\rho^a - \xi^\rho T_{\rho\mu}^a \,, \\
\delta A_\mu^m &= \xi^\rho \partial_\rho A_\mu^m + \partial_\mu \xi^\rho e_\rho^a - g A_\mu^n \partial_n \xi^\rho A_\rho^m - \xi^\rho F_{\rho\mu}^m \,.
\end{aligned} \tag{4.45}$$

We see that up to field strength terms, which vanish by the equations of motion (4.38), the deformed spin-2 transformations are correctly reproduced for e_μ^a and for A_μ^m up to scalar-dependent terms. Thus on-shell the spin-2 variations are contained in the non-abelian gauge transformations. Even though (4.43) is only on-shell equivalent to (2.7) and (2.9), the gauged diffeomorphisms are also an (off-shell) symmetry. This can be checked explicitly, but follows also from the fact that gauge transformations in general can be 'twisted' by terms proportional to the equations of motion [36]. In fact, a symmetry $\delta\phi^i$ (where ϕ^i generically denotes the fields) can be rewritten by use of so-called trivial gauge transformation,

$$\bar{\delta}\phi^i = \delta\phi^i + \Omega^{ij}E_j \ , \tag{4.46}$$

where $E_i = \delta\mathcal{L}/\delta\phi^i$, if the (space-time dependent) Ω^{ij} are anti-symmetric. If we choose in case of the Chern-Simons theory this matrix to be $\Omega_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho}\xi^\rho$ the twisted gauge transformation reads by use of $E^\mu = \epsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho}$

$$\bar{\delta}\mathcal{A}_\mu = \delta\mathcal{A}_\mu + \Omega_{\mu\nu}E^\nu = \delta\mathcal{A}_\mu + \xi^\rho\mathcal{F}_{\rho\mu} \ , \tag{4.47}$$

i.e. receives the term proportional to the field strength in (4.45).

In total we have found an action which is invariant under deformed spin-2 gauge transformations or, equivalently, under gauged diffeomorphisms, with the latter being realized as ordinary Yang-Mills gauge transformations. This has been achieved by virtue of a Chern-Simons formulation based on a centrally extended Poincaré algebra. In the next section we will turn to the problem of reconstructing the full Einstein-Hilbert action.

4.3 The full theory

Let us now discuss the problem of gauging the internal diffeomorphism algebra $\hat{v}_d$ for the scalar fields. Also for this we first have to replace the partial derivatives by the covariant ones defined in (4.21). However, in the last section we have seen that in order to get a consistent covariantisation of the Einstein-Hilbert term we have to introduce additional gauge fields $B_{\mu m}$, which gauge the shift symmetries of $SL(d+1,\mathbb{R})$. Since these shift symmetries act also on the scalars φ_m according to $\delta_\Lambda\varphi_m = -g\Lambda_m$, this implies that the covariant derivative for the latter has to be extended to

$$\mathcal{D}_\mu\varphi_m = D_\mu\varphi_m + gB_{\mu m} \ . \tag{4.48}$$

Performing this minimal substitution in (3.14), results in an action of the form

$$\begin{aligned} S_g = \int d^3x d^d y\, e\Big(&- R_3^{\text{cov}} - \frac{1}{2} g e^{-1}\varepsilon^{\mu\nu\rho}B_{\mu m}F_{\nu\rho}^m + \phi^{-2}D^\mu\phi D_\mu\phi \\ &+ \frac{1}{2}\phi^{-2}G^{mn}(\phi)\mathcal{D}^\mu\varphi_m\mathcal{D}_\mu\varphi_n + \frac{1}{2}(\phi_\alpha^m D^\mu\phi_m^\gamma)(\phi_\gamma^n D_\mu\phi_n^\alpha) \\ &- \frac{1}{2}G^{mn}(\phi)D^\mu\phi_m^\beta D_\mu\phi_{n\beta} + \mathcal{L}_{\text{gauge}}\Big) \ . \end{aligned} \tag{4.49}$$

Here R_3^{cov} denotes the (2+1)-dimensional Einstein-Hilbert term, covariantized with respect to the local $\hat{v}_d$ symmetry, as introduced in (4.37). Moreover, we added the Chern-Simons term for the gauge vectors (setting $\alpha = 2$ in (4.37)), which is required according to the analysis in the previous section. Finally, as a precaution we supplemented the action by additional couplings $\mathcal{L}_{\text{gauge}}$ which, by the experience from S^1 theories and gauged supergravities [12, 13], are expected to appear.

In fact, facing the problem whether (4.49) is invariant under the g-deformed spin-2 transformations introduced in 4.1, we have to conclude that this is in general not the case, since the consistency problems have been resolved only for the topological subsector consisting of gravitational and gauge fields. This in turn is the reason that the transformation rules and couplings have to be extended further. We do not aim to determine all possible couplings systematically in this paper, but instead prove that those resulting from a direct Kaluza-Klein analysis fit exactly into a theory of the form given in (4.49). More precisely, we show that upon choosing $\mathcal{L}_{\text{gauge}}$ in (4.49) as determined by the Kaluza-Klein approach in appendix B one gets an action, which is on-shell equivalent to the full Kaluza-Klein theory and whose invariance under spin-2 transformations can be traced back to the invariance of the original Kaluza-Klein theory.

First we have to show that (4.49) with the Chern-Simons gauge fields is equivalent to the Kaluza-Klein theory containing Yang-Mills terms. This can be seen in the same way as in gauged supergravity, following [34, 35]. In fact, varying (4.49) with respect to $B_{\mu m}$ results in

$$\mathcal{D}_\mu \varphi_m = \frac{1}{2}\phi^2 G_{mn}\varepsilon_{\mu\nu\rho}F^{\nu\rho\, n} \ , \tag{4.50}$$

which in the ungauged limit $g \rightarrow 0$ reduces to the standard duality relation (A.2). This in turn implies that the equations of motion for (4.49) are equivalent to those in the Yang-Mills gauged form (B.6), which can be most easily seen by choosing the gauge fixing $\varphi_m = 0$ and then integrating out $B_{\mu m}$. Finally, we know from (4.40) that varying (4.49) with respect to ω_μ^a results in the $\hat{v}_d$ covariantized torsion constraint. Using the latter to solve for ω_μ^a in terms of e_μ^a and A_μ^m, one gets an $\hat{v}_d$ covariantized Einstein-Hilbert term, which coincides exactly with that appearing upon direct dimensional reduction. Indeed, in appendix B we prove that the Kaluza-Klein theory has the required form, where $\mathcal{L}_{\text{gauge}}$ is given by

$$\begin{aligned}\mathcal{L}_{\text{gauge}} = &-\frac{1}{2}g^2\phi^{-2}G^{mn}(\phi)(e_a^\nu D_m e_\nu^b)(e_b^\mu D_n e_\mu^a) - \frac{1}{2}g^2\phi^{-2}G^{mn}(\phi)g^{\mu\nu}D_m e_\mu^a D_n e_{\nu a} \\ &- g^2\phi^{-2}G^{mn}(e^{a\mu}D_m e_{\mu a})(e^{b\nu}D_n e_{\nu b}) + g^2\phi^{-2}R(\phi) - \frac{1}{2}gF^{ab\, m}e_{[a}^\nu D_m e_{\nu b]} \ .\end{aligned} \tag{4.51}$$

Here $R(\phi)$ denotes the Ricci scalar computed with respect to the internal vielbein ϕ_m^α in the standard fashion. The terms quadratic in $D_m e_\mu^a$ are mass terms for the spin-2 fields, while $R(\phi)$ is a scalar potential. With (4.51) we note that in the decompactification limit $g \rightarrow 0$ the theory indeed reduces to (3.11).

After we have proven that (4.49) is equivalent to the D-dimensional Einstein-Hilbert action, we can conclude that it admits the gauged diffeomorphisms discussed in 4.1 (after adding the scalar-dependent contribution in $\delta_\xi A_\mu^m$) as a local symmetry. However, in contrast to the Chern-Simons part of (4.49), the symmetry on the remaining couplings is far from being manifest. The difficulty in analyzing this symmetry is due to the fact that the original theory is in a 2$^{\text{nd}}$ order form, in which the spin connection ω_μ^a and the dual vector $B_{\mu m}$ do not appear as independent fields, but are determined by their equations of motion, while the scalars φ_m are altogether absent. In the following let us therefore briefly comment on the different realization of symmetries in a 1^{st} order and a 2^{nd} order formulation.

Concerning the problem to find an independent symmetry variation for the spin connection, we note that ω_μ^a, when expressed in terms of e_μ^a and A_μ^m, does not transform simply as in the 1^{st} order Chern-Simons formulation in (4.43), but in a highly non-trivial manner. The latter fact can be traced back to the same origin as the non-covariance of $\hat{v}_d$-covariant derivatives under gauged diffeomorphisms discussed in sec. 4.1. It is nevertheless always possible for a given 2^{nd} order action with a certain local symmetry and an on-shell equivalent 1^{st} order action to find a corresponding local (off-shell) symmetry on the 1^{st} order fields (for a systematic account see [36]). But, it has to be taken into account that these variations in general cannot just be determined by applying the 2^{nd} order variation to, in our case, $\omega_\mu^a(e, A)$, but they receive additional contributions which, however, vanish on-shell (see, e.g., eq. (2.7) in ref. [36]). Even more, the 1^{st} order formulation is not unique since trivial gauge transformations as in (4.46) can be added, which in turn can simplify the expressions significantly. One may compare with the situation in supergravity. Applying naively the mentioned results of [36] in order to get a 1^{st} order formulation of pure $N = 1$ supergravity in $D = 4$ [37] results in a rather intricate supersymmetry variation for the spin connection. Only after a trivial gauge transformation they take a much simpler, namely supercovariant form, in the sense that the variations do not contain derivatives of the supersymmetry parameter. Similarly, one may hope to find a true 1^{st} order formulation of (4.49), which treats both ω_μ^a and $B_{\mu m}$ as independent fields, and which takes advantage of the Chern-Simons formulation of sec. 4.2. We will leave this for future work.

5 Conclusions and Outlook

In this paper we further analyzed the local spin-2 and global hidden symmetries that appear in Kaluza-Klein theories once all massive modes are taken into account, generalizing [5] to the case of generic internal manifolds. We found that in the unbroken phase the hidden rigid symmetry group $SL(D-2, \mathbb{R})$, that appears by reducing D-dimensional gravity on a torus to three dimensions, is enhanced to the infinite-dimensional current group associated to the internal manifold K_d. Moreover, the diffeomorphism algebra of the internal manifold, generalizing the Witt algebra in the case of a circle reduction, shows up as a global symmetry. The broken phase in turn results from gauging of $\hat{v}_d$ and a certain subalgebra of $T^d sl(d+1, \mathbb{R})$. We proved that the spin-2 and spin-1 fields can be incorporated into an action of Chern-Simons form and we gave the underlying gauge algebra explicitly in case of a torus.

Even though our analysis was restricted to a torus as far as the mode expansion or gauge algebra is concerned, we can immediately conclude that the presented picture holds more generally. In fact, since the action can be entirely rewritten in terms of fields depending on the internal coordinates y^m, without any reference to the topology of a torus, the unbroken phase (3.14) as well as the Chern-Simons action (4.37) exist for any internal manifold K_d and have the required symmetries. For instance, since the Chern-Simons action is gauge invariant for any K_d, we know that the analogue of the gauge algebra (4.33) has to exist and can be given explicitly, e.g., simply by expanding the gauge transformations (4.43) in harmonics of K_d and reading off the Lie algebra from the homogenous terms. Similarly, for the global symmetry algebra in the unbroken phase the semi-direct product between $diff(K_d)$ and $K_d sl(d+1,\mathbb{R})$ is completely determined by (3.18) and (3.19). In fact, once all Kaluza-Klein modes are taken into account, the resulting theory is basically independent of the internal manifold. The latter is of crucial importance only if truncations are considered, and one may hope that the knowledge of the corresponding gauge algebra allows a systematic analysis of consistent truncations as an algebraic problem (compare [38]).

Let us stress again that we have essentially provided a reformulation of pure gravity in any dimension D, since no truncations were involved. However, the duality transformations specific for $D = 3$ were still possible, and so the physical degrees of freedom in (4.49) are described by the scalars of a gauged non-linear σ-model, while the 'kinetic terms' of the graviton modes are given by a topological Chern-Simons action. This reformulation may therefore enlighten the meaning of hidden symmetries in terms of the original, higher-dimensional theory in that the latter can be viewed as a deformation (in the sense of a gauging) of a theory which has these symmetries.

This work can be extended into various directions. First of all, a true 1$^{\text{st}}$ order formulation as discussed in the previous section would clearly be desirable in order to analyze more systematically which kind of couplings are allowed by gauged diffeomorphisms. The latter is particularly important for the construction of theories which are not expected to have a formulation as a Lorentz-invariant gravitational theory, like the 12-dimensional theory proposed in [39–41]. Second, the AdS and supersymmetric extension would be interesting [42], e.g. applied to 11-dimensional supergravity.

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A Appendix: Non-linear realisation of $SL(d+1,\mathbb{R})$

It is well known that Kaluza-Klein reduction of (super-)gravities leads to 'hidden' symmetries [27–29]. The simplest example is the so-called Ehlers group, which appears upon reducing four-dimensional gravity to $D = 3$, or equivalently, by considering Einstein gravity with one space-like isometry. More specifically, after dualization into a scalar, the Kaluza-Klein vector spans together with the dilaton the σ-model coset space $SL(2,\mathbb{R})/SO(2)$. This yields the isometry group $SL(2,\mathbb{R})$ as rigid invariance group, which in turn does not have an obvious higher-dimensional ancestor. This phenomenon generalizes to the case of arbitrary torus reductions: Reducing Einstein gravity on T^d to $D = 3$ yields upon dualization a σ-model with target space $SL(d+1,\mathbb{R})/SO(d+1)$, which can be seen as follows.

In order to dualize the Kaluza-Klein vectors A_μ^m into scalars φ_m we enforce as usual the Bianchi identity by means of Lagrange multipliers, i.e. the Yang-Mills term in (2.5) gets replaced by the new action

$$\mathcal{L}'(F,\varphi) = -\frac{1}{4}\phi^2 G_{mn}(\phi) F^{\mu\nu\, m} F^n_{\mu\nu} - \frac{1}{2}\varphi_m \varepsilon^{\mu\nu\rho}\partial_\mu F^m_{\nu\rho} \; . \tag{A.1}$$

Varying with respect to $F^m_{\mu\nu}$ yields the duality relation

$$\partial_\mu \varphi_m = \frac{1}{2}\phi^2 G_{mn}(\phi)\varepsilon_{\mu\nu\rho}F^{\nu\rho m} \; . \tag{A.2}$$

Integrating out $F^m_{\mu\nu}$ and combining with the dimensionally reduced action in (2.5) implies

$$\begin{aligned} S = \int d^3x\, e\big[-R^3(e) + g^{\mu\nu}(\phi^{-2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\phi^{-2}G^{mn}(\phi)\partial_\mu\varphi_m\partial_\nu\varphi_n \\ + \frac{1}{2}\phi^m_\alpha\partial_\mu\phi^\gamma_m\phi^n_\gamma\partial_\nu\phi^\alpha_n - \frac{1}{2}G^{mn}(\phi)\partial_\mu\phi^\beta_m\partial_\nu\phi_{n\beta})\big] \; . \end{aligned} \tag{A.3}$$

To see that this action carries indeed a coset space structure, let us briefly recall the Lie algebra of $SL(d+1,\mathbb{R})$. It is convenient to start from the subalgebra $sl(d,\mathbb{R})$. The latter is a (d^2-1)-dimensional algebra which is spanned by the generators $K^a{}_b$, $a,b=1,..,d$, satisfying $K^a{}_a = 0$. The Lie algebra reads

$$[K^a{}_b, K^c{}_d] = \delta^a_d K^c{}_b - \delta^c_b K^a{}_d \; . \tag{A.4}$$

An explicit representation by traceless matrices is given by

$$(K^a{}_b)^m{}_n = \delta^a_n\delta^m_b - \frac{1}{d}\delta^a_b\delta^m_n \; . \tag{A.5}$$

The elements of $sl(d+1,\mathbb{R})$ can then be written as

$$\hat{K}^a{}_b = \begin{pmatrix} K^a{}_b & 0 \\ 0 & 0 \end{pmatrix} , \quad \hat{e}_a = \begin{pmatrix} 0 & e_a \\ 0 & 0 \end{pmatrix} , \quad \hat{f}^a = \begin{pmatrix} 0 & 0 \\ f^a & 0 \end{pmatrix} , \quad \hat{e} = \begin{pmatrix} -\mathbf{1}_d & 0 \\ 0 & d \end{pmatrix} , \tag{A.6}$$

where the components of the column and row vectors are defined by

$$(e_a)^m = \delta^m_a \; , \qquad (f^a)_m = \delta^a_m \; . \tag{A.7}$$

The Lie algebra of $sl(d+1,\mathbb{R})$, extending the $sl(d,\mathbb{R})$ subalgebra (A.4), can then be easily computed and turns out to be (after dropping the hats)

$$\begin{aligned} &[K^a{}_b, e_c] = \delta^a_c e_b - \frac{1}{d}\delta^a_b e_c \; , \qquad [K^a{}_b, f^c] = -\delta^c_b f^a + \frac{1}{d}\delta^a_b f^c \; , \\ &[e_a, f^b] = K^b{}_a - \frac{1}{d}\hat{e}\delta^b_a \; , \quad [K^a{}_b, \hat{e}] = 0 \; . \\ &[e_a, \hat{e}] = (d+1)e_a \; , \qquad [f^a, \hat{e}] = -(d+1)f^a \; , \\ &[e_a, e_b] = 0 \; , \qquad [f_a, f_b] = 0 \; . \end{aligned} \tag{A.8}$$

Next we can turn to the construction of the non-linear σ-model with target space $SL(d+1,\mathbb{R})/SO(d+1)$. The scalar fields will be described by a group-valued matrix

$$\mathcal{V} = \left(\begin{array}{cc} \phi_m^\alpha & -\phi^{-1}\varphi_m \\ 0 & \phi^{-1} \end{array} \right) , \tag{A.9}$$

where we have fixed some of the $SO(d+1)$ symmetry to choose a triangular gauge. Then one can compute the Lie-algebra-valued current

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} = \left(\begin{array}{cc} \phi_\alpha^m \partial_\mu \phi_m^\beta & -\phi^{-1}\phi_\alpha^m \partial_\mu \varphi_m \\ 0 & -\phi^{-1}\partial_\mu \phi \end{array} \right) . \tag{A.10}$$

The σ-model action can now be defined by decomposing this current into compact and non-compact parts, i.e. by decomposing it into anti-symmetric and symmetric matrices. Denoting the non-compact part by brackets $[,]$, the resulting action reads

$$\begin{aligned} \mathcal{L}_{\text{coset}} &= g^{\mu\nu}\text{Tr}\left([\mathcal{V}^{-1}\partial_\mu\mathcal{V}][\mathcal{V}^{-1}\partial_\nu\mathcal{V}]\right) \\ &= \phi^{-2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}\phi^{-2}G^{mn}(\phi)\partial^\mu\varphi_m\partial_\mu\varphi_n + \frac{1}{2}(\phi_\alpha^m\partial^\mu\phi_m^\beta)(\phi_\beta^n\partial_\mu\phi_n^\alpha) \\ &\quad -\frac{1}{2}G^{mn}(\phi)\partial^\mu\phi_m^\beta\partial_\mu\phi_{\beta n} \,. \end{aligned} \tag{A.11}$$

Looking back to (A.3) one infers that this coincides (after coupling to gravity) with the dimensionally reduced action after dualisation. By construction, this σ-model action has a non-linear rigid $SL(d+1,\mathbb{R})$ symmetry. The group action on the fields corresponding to this enhanced symmetry will be determined in the following.

To start with, we remind the reader that for a coset space G/H the G acts rigidly on a group element like in (A.9) by left multiplication, while the maximal compact subgroup H acts by local right multiplication. Infinitesimally, it reads in the given case

$$\delta\mathcal{V} = \hat{g}\mathcal{V} - \mathcal{V}\hat{h}(x) \,, \tag{A.12}$$

where $\hat{g} \in sl(d+1,\mathbb{R})$ and $\hat{h}(x) \in so(d+1)$. With (A.12) one finds that the $sl(d,\mathbb{R})$ acts on the fields linearly as

$$\delta_\lambda(K)\phi_m^\alpha = \lambda^a{}_b(K^b{}_a)_m{}^n\phi_n^\alpha \,, \qquad \delta_\lambda(K)\varphi_m = \lambda^a{}_b(K^b{}_a)_m{}^n\varphi_n \,. \tag{A.13}$$

In contrast, the symmetries generated by e_a act non-linearly as a shift,

$$\delta_\lambda(e)\phi_m^\alpha = 0 \,, \qquad \delta_\lambda(e)\varphi_m = -\lambda_m \,, \tag{A.14}$$

while the $\hat{e}$ transformations act in accordance with (A.8) as rescalings,

$$\delta_\lambda(\hat{e})\phi_m^\alpha = \lambda\phi_m^\alpha \,, \qquad \delta_\lambda(\hat{e})\phi = d\lambda\phi \,, \qquad \delta_\lambda(\hat{e})\varphi_m = (d+1)\lambda\varphi_m \,. \tag{A.15}$$

For the variations induced by the f^a one has to take into account that it is necessary to add a compensating local $SO(d+1)$ transformation in order to restore the triangular gauge in (A.9). Choosing for this the transformation parameter $\xi^\alpha = \phi\lambda^m\phi_m^\alpha$ (where λ^m parametrizes the rigid transformation), one finds correspondingly the non-linear group action

$$\delta_\lambda(f_a)\phi_m^\alpha = (\lambda^l\varphi_m)\phi_l^\alpha \,, \qquad \delta_\lambda(f_a)\varphi_m = (\lambda^l\varphi_l)\varphi_m \,. \tag{A.16}$$

B Explicit reduction without truncation

In this appendix we compute the 'dimensionally' reduced action, i.e. in Yang-Mills form, directly from the D-dimensional Einstein-Hilbert term

$$S_{\rm EH} = -\int d^D x\, E E_A^M E_B^N \big(\partial_M \omega_N{}^{AB} - \partial_M \omega_N{}^{AB} + \omega_M{}^{AC}\omega_{NC}{}^B - \omega_N{}^{AC}\omega_{MC}{}^B\big)\ . \tag{B.1}$$

The spin connection in flat indices is defined in terms of the coefficients of anholonomity as

$$\omega_{ABC} = \frac{1}{2}\left(\Omega_{ABC} - \Omega_{BCA} + \Omega_{CAB}\right)\ , \qquad \Omega_{AB}{}^C = 2E_{[A}^M E_{B]}^N \partial_M E_N^C\ . \tag{B.2}$$

This gives rise to the following components

$$\begin{aligned}
\Omega_{abc} &= 2\phi e_{[a}^{\mu} e_{b]}^{\nu} D_\mu e_{\nu c} + 2\eta_{c[a} e_{b]}^{\nu} D_\nu \phi\ ,\\
\Omega_{ab\alpha} &= \phi^2 F_{ab\ \alpha} := \phi^2 e_a^\mu e_b^\nu F_{\mu\nu}^n \phi_{n\alpha}\ ,\\
\Omega_{\alpha bc} &= g\phi_\alpha^m e_b^\nu D_m e_{\nu c}\ ,\\
\Omega_{\alpha\beta\gamma} &= 2g\phi_{[\alpha}^m \phi_{\beta]}^n \partial_m \phi_{n\gamma}\ ,\\
\Omega_{\alpha\beta c} &= 0\ ,\\
\Omega_{a\alpha\beta} &= \phi\phi_\alpha^m e_a^\mu D_\mu \phi_{m\beta}\ .
\end{aligned} \tag{B.3}$$

Here we have introduced an internal covariant derivative

$$D_m e_\mu^a = \partial_m e_\mu^a - (\phi^{-1}\partial_m\phi) e_\mu^a\ , \tag{B.4}$$

which transforms covariantly under local $\hat{v}_d$ transformations (albeit in a different representations as e_μ^a):

$$\delta_\xi(D_m e_\mu^a) = g\xi^n \partial_n (D_m e_\mu^a) + g\partial_m \xi^n D_n e_\mu^a + g\partial_n \xi^n D_m e_\mu^a\ . \tag{B.5}$$

Inserting (B.3) into the Einstein-Hilbert action (B.1) and dropping total derivatives results in

$$\begin{aligned}
S_{\rm EH} = \int d^3x d^d y\, e\big[&- R_3^{\rm cov}(e) - \frac{1}{4}\phi^2 G_{mn}(\phi) F^{\mu\nu\, m} F_{\mu\nu}^n + \phi^{-2} D^\mu\phi D_\mu \phi\\
&+ \frac{1}{2}(\phi_\alpha^m D^\mu \phi_m^\gamma)(\phi_\gamma^n D_\mu \phi_n^\alpha) - \frac{1}{2} G^{mn}(\phi) D^\mu \phi_m^\beta D_\mu \phi_{n\beta}\\
&+ g^2\phi^{-2} R(\phi) - \frac{1}{2} g F^{ab\, m} e_{[a}^\nu D_m e_{\nu b]}\\
&- \frac{1}{2} g^2 \phi^{-2} G^{mn}(\phi)(e_a^\nu D_m e_\nu^b)(e_b^\mu D_n e_\mu^a)\\
&- \frac{1}{2} g^2 \phi^{-2} G^{mn}(\phi) g^{\mu\nu} D_m e_\mu^a D_n e_{\nu a}\\
&- g^2 \phi^{-2} G^{mn}(e^{a\mu} D_m e_{\mu a})(e^{b\nu} D_n e_{\nu b})\big]\ .
\end{aligned} \tag{B.6}$$

As claimed in the main text, the action appears in form which is manifestly invariant under local $\hat{v}_d$ transformations. Moreover, it contains the spin-2 mass terms and the scalar potential given in (4.51). Let us finally note that even though the invariance of (B.6) under local $SO(1,2) \times SO(d)$ transformations is not obvious, it can be checked explicitly. In particular, it turns out that, quite surprisingly, $R_3^{\rm cov}$ is not invariant under all local $SO(1,2)$ transformations, but its variation cancels against the variation of the term proportional to F^{abm}.

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